Problem Set 3
More Exercises on the Polynomial Hierarchy PH
CSCI 6114 Fall 2026

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Released: September 08, 2026
Due: Wednesday September 16, 2026

Exercises

1. (Complements and oracles)

   (a) Show that, for any oracle X, P^X is closed under complement, that is, $\mathsf{P}^X = \mathsf{coP}^X$. In particular, $\mathsf{P}^{\mathsf{NP}} = \mathsf{coP}^{\mathsf{NP}}$.

   (b) Show that $\mathsf{P}^{\mathsf{NP}} = \mathsf{P}^{\mathsf{coNP}}$.

   (c) Show that if $\mathsf{NP} = \mathsf{P}^{\mathsf{NP}}$, then $\mathsf{NP} = \mathsf{coNP}$.

2. (Oracle characterization of PH)

   (a) Show that $\mathsf{P}^{\mathsf{NP}} \subseteq \Sigma_2\mathsf{P}$. Use the fact that P^{NP} is closed under complement to conclude that $\mathsf{P}^{\mathsf{NP}} \subseteq \Sigma_2\mathsf{P} \cap \Pi_2\mathsf{P}$.

   (b) Show that $\Sigma_2\mathsf{P} = \mathsf{NP}^{\mathsf{NP}}$. *Hint:* the idea from part (a) may be useful.

   (c) (Read this one, take on faith in class, do outside of class.) More generally, show that $\Sigma_k\mathsf{P} = \mathsf{NP}^{\Sigma_{k-1}\mathsf{P}} = \Sigma_{k-1}\mathsf{P}^{\mathsf{NP}}$ and $\Pi_k\mathsf{P} = \mathsf{coNP}^{\Pi_{k-1}\mathsf{P}}$. This is called the *oracle characterization* of PH (since it can be used to give an alternative, equivalent, oracle-based definition of $\Sigma_k\mathsf{P}$).

3. Use the oracle characterization of PH to give an alternative (arguably simpler) proof that if $\Sigma_k\mathsf{P} = \Sigma_{k+1}\mathsf{P}$, then $\mathsf{PH} = \Sigma_k\mathsf{P}$.

4. Show that $\mathsf{NP} \cup \mathsf{coNP} \subseteq \mathsf{P}^{\mathsf{NP}}$. Observe that that proof and the proof of Exercise 2a above both relativize, to give a simple proof that $\mathsf{\Sigma_k P} \cup \mathsf{\Pi_k P} \subseteq \mathsf{P}^{\mathsf{\Sigma_k P}} \subseteq \mathsf{\Sigma_{k+1} P} \cap \mathsf{\Pi_{k+1} P}$ for all $k \geq 0$.

   Open question: (If you know or look up what the arithmetic hierarchy is from computability theory) Is there an oracle relative to which "PH looks like AH", in the following sense? In the arithmetic hierarchy, we have $\mathsf{\Sigma^0_k} \cup \mathsf{\Pi^0_k} \subsetneq \mathsf{COMP}^{\mathsf{\Sigma^0_k}} = \mathsf{\Sigma^0_{k+1}} \cap \mathsf{\Pi^0_{k+1}}$ for all $k \geq 0$ (note the *strict* containment for the first, and equality for the second!). Is there an oracle X such that

$$\mathsf{\Sigma_k P}^X \cup \mathsf{\Pi_k P}^X \subsetneq \mathsf{P}^{\mathsf{\Sigma_k P}^X} = \mathsf{\Sigma_{k+1} P}^X \cap \mathsf{\Pi_{k+1} P}^X$$

   for all $k \geq 0$?

Exercises on complete problems in PH

Given a complexity class $\mathcal{C}$, we say a problem A is *$\mathcal{C}$-hard* (under many-one reductions) if for every $B \in \mathcal{C}$, we have $B \leq_m^p A$, and we say that A is $\mathcal{C}$-complete if furthermore A is in $\mathcal{C}$.

5. Show that if PH has a complete problem, then PH collapses.

6. We define the decision problem Σ_k*CIRCUIT-SAT* as follows:

   Σ_k **CIRCUIT-SAT**
   Input: A Boolean circuit $\varphi(x_1, \ldots, x_m)$, together with a partition of $\{1, \ldots, m\}$ into k subsets $S_1, \ldots, S_k$.
   Decide: It is the case that $\exists \vec{y} \forall \vec{z} \cdots (\exists/\forall \vec{w}) \varphi(\vec{y}, \vec{z}, \ldots, \vec{w}) = 1$, where $\vec{y} = \vec{x}|_{S_1}, \vec{z} = \vec{x}|_{S_2}, \ldots, \vec{w} = \vec{x}|_{S_k}$, and the final quantifier is $\exists$ if k is odd and $\forall$ if k is even.

   Note 1: these are *not* "$\exists^p$"-style quantifiers, and that each vector $\vec{y}, \vec{z}, \ldots, \vec{w}$ is a vector of Boolean variables. The decision problem is to decide whether the quantified mathematical statement is true or false (note: the question is *not* satisfiable vs unsatisfiable, since all variables are quantified, but literally a true statement or a false statement).

   Note 2: CIRCUIT-SAT is the same as Σ_1*CIRCUIT-SAT*. (That is, satisfiable unquantified circuits are in essence the same as true statements that are $\exists$-quantified circuits.)

   Question. Show that for any $k \geq 1$, Σ_k*CIRCUIT-SAT* is $\mathsf{\Sigma_k P}$-complete. (It's also true for $k = 0$, but for somewhat trivial reasons.)

Hint: Use the idea of the proof that $\mathsf{P} \subseteq \mathsf{P/poly}$ from the first set of exercises.

(Foreshadowing: when we get to PSPACE, we will see that a related problem, Totally Quantified Boolean Formulas, or TQBF, is PSPACE-complete. TQBF is just like $\Sigma_k CIRCUIT\text{-}SAT$ except that there is no limit placed on how many quantifier alternations there can be.)

Resources

- Schöning & Pruim, *Gems of TCS*, Ch. 16. This is a nice, brief overview of what we've done in class, plus introduces the Boolean Hierarchy (what you get by taking Boolean combinations—AND, OR, NOT—of languages in NP) and shows that if BH collapses to any level, then PH collapses to its second level. And has additional nice references.
- Schaefer and Umans gave a list of many problems that are complete for the second (and a few higher) level of PH in a series of two papers.
- Du & Ko Ch. 3
- Arora & Barak Ch. 5
- Hemaspaandra & Ogihara, *Complexity Theory Companion*, Appendix A.4.1. Quick "cheat sheet"-style definitions.
- Homer & Selman §7.4